EXPLORING CRITICAL ASPECTS AND MATHEMATICAL COMMUNICATION IN ALGEBRA

Constanta Olteanu and Lucian Olteanu
Linnaeus University, Faculty of Technology, Institution of Mathematics, Sweden

While the concept of function has been extensively studied by researchers worldwide, there has been limited research conducted on this topic in Sweden. This paper aims to fill this gap by examining the significance of critical aspects and their impact on mathematical communication. The study analyzes the content presented to students regarding the concept of function, both in the classroom and through textbooks, and compares it to the actual learning outcomes. Data for the analysis was collected from two classes, involving two teachers and 45 students, including video recordings of lessons and tests. The analytical framework incorporates concepts from variation theory. The findings highlight that effective communication in the classroom hinges on considering critical aspects as the foundation of students' learning. In this study, the argument of functions is identified as a crucial aspect in understanding functions.

INTRODUCTION

The concept of function is of great importance in mathematics education, but its complexities have received limited attention within the Swedish context. Some researchers have focused on students' conceptions, difficulties, and errors related to this concept, while Sfard's (1991) process-object perspective has provided a theoretical framework for understanding algebraic function representations from operational and structural perspectives. However, there are arguments, such as those put forth by Artigue (1996), that this perspective fails to explain how students struggle to recognize the equivalence of functions with different operational processes. Researchers have highlighted the pedagogical potential of using specific examples to facilitate discussions about various aspects of functions and the intricacies of quantitative relationships. Unfortunately, the field of mathematical communication, encompassing what students learn about function concepts and their outcomes, has been insufficiently addressed. Many teachers tend to interpret function notation idiomatically, describing a rule for modeling scenarios instead of using it to represent a varying quantity (Musgrave & Thompson, 2014). This paper aims to fill this gap by applying Marton's (2015) variation theory. It explores students' conceptualizations of function notation, aiming to deepen understanding of complexities in mathematics education and highlight critical aspects essential for educators.

THEORETICAL ASSUMPTIONS

The variation theory posits that learning entails experiencing something in a qualitatively novel and more profound manner. Among its core tenets, variation theory emphasizes the crucial role of students experiencing different aspects of the learning object. Olteanu (2016) considers an object of learning as a part of an event that is formed in the communication between teacher and students. In order for students to effectively understand the concept of function, it is essential for them to grasp the comprehensive nature of functions, including their components and relationships. This involves recognizing the argument, the function's operation on the argument, and the resulting outcomes. By emphasizing different components, exploring various aspects, and combining them in different ways, students can attribute new meanings to each aspect and gain a holistic interpretation of the concept. However, if these aspects are not given sufficient attention, they remain critical for students' learning (Marton, 2015). It is important to note that critical aspects should not be confused with difficulties, errors, or obstacles. Critical aspects lie between errors, difficulties, and obstacles and serve as conditions for progress and learning. In this article, we define a critical aspect as the ability to discern certain aspects of the object of learning, whether it pertains to the whole, the constituent parts, the relationships between parts, transformations between parts, or the relations between different mathematical concepts. Inadequate aspects that hinder successful implementation in other contexts become more challenging to distinguish. If these aspects persist, they are identified as critical aspects. It is important to emphasize that critical aspects do not denote a lack of knowledge but rather the outcome of knowledge. In mathematics, a critical aspect may be valid in one field but inadequate in another. In this study, critical aspects are defined as the ability to discern specific aspects of the learning object, whether it be the whole, its constituent parts, relationships, transformations, or connections between mathematical concepts. These critical aspects may vary in their validity across different fields. Learning is examined through the lenses of the enacted object of learning (what is observed in the classroom) and the lived object of learning (how students understand the object of learning). Effective communication, as defined in this study, is the process of assigning and conveying meaning to foster shared understanding among teachers and students. The study presented in this article focuses on the same content, but the analysis reveals that the object of learning is presented differently in the two classrooms, particularly in terms of the emphasized critical aspects. The data analysis describes what students discern based on the content presented in the classroom, the identified critical aspects, and the communication that takes place in the learning environment.

METHOD

The study was conducted in two classes from the Natural Science Program in upper secondary school in Sweden. Both classes used the same textbook, and the study involved a total of 45 students (25 males, 20 females) who were 16 years old, along with two teachers named Anna and Maria. The textbook used in this study is the most chosen for use in various schools. The teachers taught the same mathematics course. Data collection consisted of five steps: the students took a diagnostic test at the beginning of the course; the lessons were recorded on video; the students took two tests during the course and a diagnostic test after the course; eight students (four in each class) were selected for an individual session, including a post-test with tasks related to concepts they needed to further develop, and an interview; the teachers reviewed the video recordings, analyzed the students' tests, and determined areas in which each student could improve their knowledge. The analysis presented in this article is based on the video-recorded lessons (12 lessons in each class) and the students' performance in a pre-test (Test 1) and a post-test (Test 2). Since both teachers taught the same content using the same textbook, it was possible to identify and describe differences in their teaching approaches. Additionally, by comparing the students' solutions to the same exercises in different tests, it was possible to identify and describe differences in their understanding of the content.

RESULTS

The analysis of the tests reveals that both classes of students face challenges in understanding the concept of function. However, it is noteworthy that students in Anna's class showed better performance compared to those in Maria's class, as indicated in Table 1.

The students' respons	Percentage of students in Maria's class		Percentage of students in Anna's class	
	algebraic (1; 2)	graphic (3;4)	algebraic (1; 2)	graphic (3;4)
The function arguments are identified with a given number or with a number on the x-axis	60 %	76 %	80 %	79 %
The function arguments are confused with the function value for this argument	20 %	24 %	5 %	14 %
The function arguments are maintained although there is a given value for x	15 %	0 %	10 %	0 %
Don't solve the tasks	5 %	0 %	5 %	7 %

Table 1. Variations in how the students perceive the function argument and value.

The discrepancy arises from how students perceive the function argument and value when presented with an algebraic expression and a graph. This observation became evident during the course while students were working on the following exercises: (1) Calculate $f(4)$ if $f(x) = 12 + x$; (2) Calculate $f(-5)$ if $f(x) = 32 - x^2$. In the third and fourth tests, students encountered a similar question regarding the value of $f(2)$. However, unlike the previous questions that were presented algebraically, this question was represented graphically. In these exercises, the functions were presented explicitly with their arguments, and the questions aimed to identify the corresponding function values. The results indicate that a higher percentage of students in Anna's class were able to discern the function arguments and understand the relationship between the arguments and the function values, both algebraically and graphically. To better understand why students in Maria's class struggled with discerning function arguments and values, it is necessary to analyze the enacted object of learning.

In both analyzed lessons, the focus was on the notation "$y, f(x)$ or $y = f(x)$" for functions, but notable differences arose in how the graphical representation and corresponding notations were addressed. In Maria's class and the textbook, the representation ($y = f(x)$) was used without providing any explanation. Maria primarily focused on the procedure for determining the value of the function, without delving into the meaning behind the notations. On the other hand, in Anna's class, the notations were presented separately, but with an emphasis on their shared meaning. Maria tended to use the notation "y" (implicit argument) when working with various functions that had an algebraic representation. In contrast, Anna's class and the textbook utilized the notation "$f(x)$" (explicit argument). Additionally, Maria's ambiguous use of the concept, especially when discussing zeros of a second-degree function, created challenges. For instance, she employed a system of equations to determine these zeros. In classroom communication, it becomes evident that Maria emphasizes the inclusion of second-degree algebraic expressions in a system of equations, diverging from the students' previous experience with two first-degree equations. She highlights that a solution to a system of equations entails assigning values to variables to satisfy all the equations simultaneously. In this context, the symbol "y" is employed to select an equation with two unknown variables. Furthermore, Maria emphasizes that a component of the system consists of a function and a line. Consequently, the symbol "y" assumes multiple meanings, serving as an unknown quantity, equation selector, and function indicator. This ambiguity made it challenging for students to discern whether exercises focused on functions or equations, evident when individually tackling various exercises. For instance, Viveka, a student in Maria's class, encounters difficulty in simplifying the expression $f(1 + h) - f(1)$ using the function $f(x) = kx + m$.

Viveka: I simplified it, so it's … […]

Viveka writes: $f(l + h) – f(1) = f + fh – f$ on the paper.

Maria: No, you must not do that because $l + h$ is a single number. But you have f of $1 + h$ when f of x becomes $kx + m$. That means that x is equal to $1+ h$.

Anna consistently employed the notation $f(x)$ when engaging with different functions represented algebraically. This approach provided the students with an opportunity to observe variations in how the fundamental components of a function (argument, operation on the argument, and resulting output) interconnect and contribute to the overall concept of the function. Anna emphasizes that the symbol y can be present in an equation and used to identify the coordinates (x, y) on the function graph. This introduces a variation in the function's argument, demonstrating that it can be represented explicitly as $f(x)$ to denote a function, or implicitly as $y = x^2 - 4x - 5$, where its meaning can be interpreted as either a function or an equation.

DISCUSSION AND CONCLUSION

The critical aspects identified are the function's argument and the relation between explicit and implicit arguments in the notations y and $f(x)$. The results indicate that despite classroom communication and textbook explanations, the argument of functions remains challenging for students to grasp, particularly in Maria's class compared to Anna's class. One possible reason for this disparity is Maria's use of implicit arguments without clear explanations or the presence of ambiguous relations between different components of the object of learning (function and equation). In contrast, Anna focuses on the critical aspect of the content in her communication. In Maria's class, students struggle to make sense of the signs, while in Anna's class, they can assign meaning to the signs more easily. The teacher's role is crucial in facilitating effective and natural learning experiences and identifying the key focus points related to the concept of function. Notations for functions should be unambiguous, as using the same notation for different objects hampers students' ability to discern the parts and relationships within a function. Given the popularity of the used textbook in Swedish upper secondary schools and the tendency of teachers to interpret function notation idiomatically, it is crucial to clarify the purpose and significance of different notations in problem-solving. This clarification is essential for promoting comprehension and understanding, as specified by Musgrave and Thompson (2014).

References

Artigue, M. (1996). Learning and teaching elementary analysis. In C. Alsina, J. M. Alvarez, A. M. Niss, L. Rico, & A. Sfard (Red.), *8th International Congress on Mathematics Education – selected lectures* (pp. 15–30). S.A.E.M Thalès.

Musgrave, S. & Thompson, P. W. (2014). *Function notation as idiom*. Paper presented at the Proceedings of the 38th Meeting of the International Group for the Psychology of Mathematics Education, Vancouver, BC.

Marton, F. (2015). *Necessary Conditions of Learning*. London: Routledge

Olteanu, L. (2016). *Framgångsrik kommunikation i matematikklassrummet*. Diss. Linnaeus University Press.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.